JOURNAL OF APPROXIMATION THEORY 24, 146–160 (1978)

Approximation of Meromorphic Functions by Rational Functions

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Communicated by Richard S. Varga

Received February 18, 1977

Some inequalities proved by Meinardus and Varga, and by Erdös and Reddy, on Chebyshev constants for the function $1/f$, f entire and satisfying some conditions, have been improved or extended to functions satisfying a different set of conditions.

1. INTRODUCTION

Let $f(z)$ be a transcendental entire function with $f^{(k)}(0) \geqslant 0$ for all k, and let

$$\lambda_{0,n} = \inf_{p \in \pi_n} \sup_{0<x<\infty} \left| \frac{1}{f(x)} - \frac{1}{p(x)} \right| \tag{1.1}$$

denote the Chebyshev approximation constant for $1/f$. Here π_n denotes the collection of all real polynomials of degree at most n.

For f of perfectly regular growth, Meinardus and Varga [7] have proved

THEOREM A. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be any entire function of perfectly regular growth (ρ, B) with $a_k \geqslant 0$ for all $k \geqslant 0$, and for any nonnegative integers m and n let*

$$\lambda_{m,n}^* = \inf_{\substack{p \in \pi_m \\ q \in \pi_n}} \sup_{0<x<\infty} \left| \frac{1}{f(x)} - \frac{p(x)}{q(x)} \right|$$

denote the Chebyshev constants for $1/f$. Then for any sequence $\{m(n)\}_{n=0}^{\infty}$ with $0 \leqslant m(n) \leqslant n$ for each $n \geqslant 0$,

$$\limsup_{n\to\infty} (\lambda_{m(n),n}^*)^{1/n} \leqslant 1/2^{1/\rho}. \tag{1.2}$$

Moreover,

$$\limsup_{n\to\infty} (\lambda_{0,n}^*)^{1/n} \geqslant 1/2^{2+1/\rho}. \tag{1.3}$$

0021-9045/78/0242-0146$02.00/0

In Theorems 1 and 2 of this paper we obtain inequalities for $\lambda_{0,n}$, for a class of entire functions f of infinite order. In Theorem 3 and Corollary 3.1, we consider f of any order and obtain inequalities, valid for all $n \geqslant 1$, when $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and the coefficients a_k satisfy some regularity conditions. In Corollary 3.2 we require that f be of "smooth growth." Corollary 3.3 and Theorem 4 extend Theorem A.

For some inequalities giving upper bounds to $\liminf_{n\to\infty} \lambda_{0,n}^{1/n}$ see [13] and the references given there.

In the sequel, $r > r_0$ (or $n > n_0$ or $x > x_0$) will mean that r (resp. n, x) is sufficiently large. The value r_0 (or n_0, x_0) will in general vary. A, A_1, A_2, B, α, β, c, c_1 will denote positive numbers; and e_k and l_k will denote the kth iteration of the exponential function, and logarithmic function so that $e_1(x) = e^x$, $l_1 x = \log x$ [3, p. 16].

2. Inequalities for $\lambda_{0,n}$

The required function f in Theorem 1 will depend on a given function F of order one maximal type.

Theorem 1. *Let $F(z)$ be an entire function of order one, maximal type, and of perfectly regular growth with respect to a proximate order, that is,*

$$\lim_{r\to\infty} \log M(r, F)/r^{\rho(r)} = 1, \qquad \lim_{r\to\infty} \rho(r) = 1.$$

Write

$$L(r) = r^{\rho(r)-1}, \qquad \psi(r) = \int_{r_0}^{r} \frac{dt}{tL(t)}. \tag{2.2}$$

Suppose now that

$$\lim_{r\to\infty} \frac{\psi(r)\, rL(r)}{L(e^r)} = \infty, \tag{2.3}$$

and that there exists an entire function η with nonnegative coefficients such that for all $r > r_0$,

$$A\psi^{-1}(r)\, L(\psi^{-1}(r)) < \eta'(r) < B\psi^{-1}(r)\, L(\psi^{-1}(r)) \tag{2.4}$$

for some pair (A, B) of positive numbers. Then for $f(z) = e_2(\eta(z))$

$$\lambda_{0,n} \geqslant \exp(-n/(L(n))^{1/2}) \tag{2.5}$$

for an infinity of n.

COROLLARY 1.1. *Let K be a fixed integer and $2 \leqslant k \leqslant K$. Assume the hypotheses of Theorem 1 and suppose further that*

$$\psi(r)\, rL(r) \geqslant L(e^r)\, \psi(e^r), \qquad r > r_0 . \tag{2.6}$$

Then for each function $f_k(z) = f(z) = e_k(\eta(z))$ $(2 \leqslant k \leqslant K)$

$$\lambda_{0,n} \geqslant \exp(-n/(L(n))^{1/2})$$

for an infinity of n.

Remarks. (i) Condition (2.3) assures that $\psi(r)$ is a strictly increasing unbounded function of r for $r > r_0$. Hence the inverse function $\psi^{-1}(r)$ exists on (r_0, ∞).

(ii) Condition (2.4) permits us to construct many functions f for which the conclusion (2.5) holds. For instance, we can take $f(z) = e_2(\eta(z) + \xi(z))$, where $\xi(z)$ is any entire function, with nonnegative coefficients, such that $\xi'(r) = o(\eta'(r))$.

EXAMPLE 1.2. Let F be an entire function such that

$$\log M(r, F) \sim r l_1 r l_2 r \cdots l_p r, \qquad p > 1.$$

Take $L(r) = l_1 r \cdots l_p r$, $r > e_p(1)$. Then $\psi(r) = l_{p+1} r$, by choosing r_0 in (2.2) suitably. Condition (2.6) is satisfied. Take $\eta(z) = e_{p+1}(z)$. Then $\eta'(r) = e_{p+1}(r)\, e_p(r) \cdots e_1(r)$, $\psi^{-1}(r) = e_{p+1}(r)$, $L(\psi^{-1}(r)) = e_p(r) \cdots e_1(r)$ and so (2.4) is satisfied. Now we choose $f(z) = e_{k+p+1}(z)$ $(2 \leqslant k \leqslant K)$ and get

$$\lambda_{0,n} \geqslant \exp\left(\frac{-n}{\{l_1 n \cdots l_p n\}^{1/2}}\right)$$

for an infinity of n.

In the next theorem we use the properties of logarithmico-exponential functions [3, p. 17].

THEOREM 2. *Let $L(x)$ be positive and continuous for $x > x_0$ and suppose that*

$$\psi(x) = \int_{x_0}^{x} \frac{dt}{tL(t)} \tag{2.7}$$

is an unbounded logarithmico-exponential function. Then there exists an entire function f of infinite order such that

$$\lambda_{0,n} \geqslant \exp\left(\frac{-n}{L(n))^{1/2}}\right)$$

for an infinity of n.

These two theorems extend and improve the corrected version of [2, Theorem 2]. (For the corrections, see Errata to [2].)

The next theorem gives an inequality for $\lambda_{0,n}$ valid for $n \geqslant 1$. Here f may be of any order, finite or infinite.

THEOREM 3. *Let f be an entire function defined by*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_0 > 0, a_k \geqslant 0. \tag{2.8}$$

Let $\{d_k\}_{k=1}^{\infty}$ be any strictly increasing unbounded sequence of positive numbers. Then for $n \geqslant 1$

$$\lambda_{0,n} \geqslant (a_{n+1} d_n^{n+1})/(2^{2n+2}\{f(d_n)\}^2). \tag{2.9}$$

In the following Corollary 3.1 we make a suitable choice for d_n and in Corollary 3.2 we put conditions on the asymptotic behavior of f.

COROLLARY 3.1. *Let f be defined by* (2.8). *Suppose further that $a_k \neq 0$ and write $d_k = a_{k-1}/a_k$ $(k \geqslant 1)$. Assume also that*

$$d_{k+1} > d_k, \qquad k \geqslant 1, \tag{2.10}$$

$$d_k \geqslant \left(\frac{k}{k-1}\right) d_p, \qquad p = [kc], k > n_0, \tag{2.11}$$

for some number c in (0, 1). *Then for all $n \geqslant 1$,*

$$\lambda_{0,2n-1} \geqslant \frac{A}{n^2 2^{4n}} \frac{d_1 \cdots d_n}{d_{n+1} \cdots d_{2n}}, \tag{2.12}$$

where A is a positive constant which may depend on a_0 and c.

COROLLARY 3.2. *Let f be defined by* (2.8). *Suppose that $a_k \neq 0$ and assume that $d_{k+1} > d_k = a_{k-1}/a_k$ $(k \geqslant 1)$. Suppose further the following* (*see* [6]):

There exists a positive function φ defined for all positive numbers such that;

φ' is positive and unbounded,
φ'' is positive and continuous,

and for all large x

$$\alpha \frac{\varphi'(x)}{\varphi(x)} < \frac{\varphi''(x)}{\varphi'(x)} < \beta \frac{\varphi'(x)}{\varphi(x)},$$

for some pair (α, β) *of positive numbers, and*

$$\log M(r, f) \sim \varphi(\log r).$$

Then (2.12) *holds. The constant* A *may now depend on* a_0, α, *and* β.

COROLLARY 3.3 *Let* f *be defined by* (2.8). *Let* m_0, N_0 *be the smallest positive integers such that* $m_0 \leqslant N_0$ *and* $a_k > 0$ $(k \geqslant m_0)$,

$$a_{k-1}/a_k < a_k/a_{k+1} \qquad (k > N_0).$$

Let $0 < \rho < \infty$ *and suppose that* f *is of perfectly regular growth with respect to a proximate order* $\rho(r)$, *that is,*

$$\lim_{r\to\infty} \frac{\log M(r, f)}{r^{\rho(r)}} = 1, \qquad \lim_{r\to\infty} \rho(r) = \rho.$$

Then

$$\liminf_{n\to\infty} \lambda_{0,n}^{1/n} \geqslant 1/2^{2+1/\rho}.$$

This corollary improves (1.3). Note that we have assumed here a somewhat different hypothesis than that of Theorem A. In the next theorem we assume that a_k satisfies an asymptotic relation, and extend (1.2) and (1.3) to functions f which may not be of perfectly regular growth (ρ, a).

THEOREM 4. *Let*

$$a_k \sim (1/kL(k))^{k/\rho}, \qquad k \to \infty, 0 < \rho < \infty, \tag{2.13}$$

where $L(x)$ *is any real valued function positive on* $[n_0, \infty)$ *and* $\{x \log(xL(x))\}$ *strictly convex. Suppose further that* $\lim_{x\to\infty} xL'(x)/L(x) = 0$. *Let* f *be defined by* (2.8) *and* (2.13). *Then* f *is an entire function of perfectly regular growth with respect to a proximate order* $\rho(r)$, $\lim_{r\to\infty} \rho(r) = \rho$ *and*

$$\limsup_{n\to\infty} \lambda_{0,n}^{1/n} \leqslant 1/2^{1/\rho}, \tag{2.14}$$

$$\liminf_{n\to\infty} \lambda_{0,n}^{1/n} \geqslant 1/2^{2+1/\rho}. \tag{2.15}$$

Remarks. (i) Erdös and Reddy have proved ([2, Theorem 4]; see also Errata to [2]), that if f is an entire function of finite order ρ, defined by

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{d_1 d_2 \cdots d_k}, \qquad d_{k+1} > d_k > 0, \tag{2.16}$$

then for $\epsilon > 0$ and all $n > n_0(\epsilon)$,

$$\lambda_{0,2n-1} \geqslant (d_1 \cdots d_n)/\{2^{4n} d_n^{2(\rho+\epsilon)} d_{n+1} d_{n+2} \cdots d_{2n}\}. \tag{2.17}$$

We do not restrict, in Theorem 3 or Corollaries 3.1 and 3.2, f to be of finite order. If f is of finite order ρ and lower order λ then [11]

$$\lim_{n\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log n}{\log d_n} = \begin{cases}\rho \\ \lambda\end{cases}; \tag{2.18}$$

and (2.18) shows that (2.12) gives a better inequality than (2.17) for the class of functions f, considered in Corollaries 3.1 and 3.2 and of order ρ.

(ii) The functions f, in Corollary 3.2, form a subset of the class of functions G defined by London [6]. This class includes all functions of finite nonzero order and of perfectly regular growth with respect to a proximate order, and also many functions of zero and infinite order. G includes, for instance, all functions f such that

$$\log M(r, f) \sim c e_k(c_1 \log r), \qquad k \geqslant 1.$$

EXAMPLE 4.1. Let $p > 1$, $\alpha_1, \ldots, \alpha_p$ real numbers. Choose $x_0 > 0$ so large that for $x \geqslant x_0$, $l_p(x_0) > 0$ and

$$\frac{d^2}{dx^2}\{x \log(x(l_1 x)^{\alpha_1} \cdots (l_p x)^{\alpha_p})\} > 0.$$

Let

$$\begin{aligned} L(x) &= (l_1 x)^{\alpha_1} \cdots (l_p x)^{\alpha_p}, && x \geqslant x_0 \\ L(x) &= L(x_0), && x < x_0. \end{aligned} \tag{2.19}$$

Then $L(x)$ satisfies the hypotheses of Theorem 4. Let f be defined by (2.8), (2.13), and (2.19). Then (2.14) and (2.15) hold. In particular, for the two functions

$$f(z) = 1 + \sum_{k=2}^{\infty} \left(\frac{\log k}{k}\right)^k z^k \tag{2.20}$$

and

$$f(z) = 1 + \sum_{k=2}^{\infty} \left(\frac{z}{k \log k}\right)^k, \tag{2.21}$$

we have

$$\limsup_{n\to\infty} \lambda_{0,n}^{1/n} \leqslant \tfrac{1}{2}; \qquad \liminf_{n\to\infty} \lambda_{0,n}^{1/n} \geqslant \tfrac{1}{8}.$$

These results improve [2, Examples 1 and 2, pp. 448–449]. Note also that f defined by (2.8), (2.13), and (2.19) (or f defined by (2.20) or (2.21)) is not of perfectly regular growth (ρ, a) if at least one $\alpha_j \neq 0$ [3, p. 53].

3. Proof of Theorem 1

By the properties of proximate orders, $L(r)$, and consequently $\psi(r)$, are slowly changing functions [5, 10]. Further, (2.3) shows that $\psi(r) \to \infty$ as $r \to \infty$ (cf. [4, pp. 296–297]). (It is not necessary to assume that $L(r)$ is an increasing function of r.) Hence the inverse function $\psi^{-1}(r)$ exists and is increasing for $r > r_0$, and is unbounded. Further, $L(r) \to \infty$ by hypothesis and so $\psi^{-1}(r)\, L(\psi^{-1}(r)) \to \infty$ with r.

Let η be an entire function with $\eta^{(k)}(0) \geqslant 0$ for all k and let

$$f(z) = e_2(\eta(z)). \tag{3.1}$$

Let $p \in \pi_n$ and be such that

$$\lambda_{0,n} = \sup_{0<x<\infty} \left| \frac{1}{f(x)} - \frac{1}{p(x)} \right| .$$

Suppose if possible

$$\lambda_{0,n} < \exp\left(\frac{-n}{(L(n))^{1/2}}\right) \tag{3.2}$$

for all $n > n_0$. Let $c = \frac{1}{3}$, $n > n_0$ and choose r_1 such that

$$\eta(r_1) = l_1\left(\frac{cn}{(L(n))^{1/2}}\right). \tag{3.3}$$

From (3.2) and (3.3) we have

$$\begin{aligned} \left| \frac{1}{p(r_1)} \right| &\geqslant \exp\left(\frac{-cn}{(L(n))^{1/2}}\right) - \exp\left(\frac{-n}{(L(n))^{1/2}}\right) \\ &> \frac{1}{2}\exp\left(\frac{-cn}{(L(n))^{1/2}}\right), \qquad n > n_0. \end{aligned} \tag{3.4}$$

Let $r_2 = r_1(1 + \delta/2)$, where $\delta = c^2/2L(n)$. Then by an inequality from Remes [9], we have for $n > n_0$,

$$| p(r_2)| \leqslant \frac{1}{2}\exp\left(\frac{2nc}{(L(n))^{1/2}}\right). \tag{3.5}$$

Now $f(r_2) > 2\,|\,p(r_2)|$ if

$$\eta(r_2) > l_1\left(\frac{2cn}{(L(n))^{1/2}}\right). \tag{3.6}$$

Note that

$$\begin{aligned}\eta(r_2) &> \eta(r_1) + (r_2 - r_1)\,\eta'(r_1)\\ &= l_1\left(\frac{cn}{(L(n))^{1/2}}\right) + \frac{r_1\delta}{2}\,A\psi^{-1}(r_1)\,L(\psi^{-1}(r_1)).\end{aligned} \tag{3.7}$$

By (2.4) we have for $r > r_0$,

$$(A/2)\,\psi^{-1}(r) < \eta(r) < 2B\psi^{-1}(r).$$

Hence

$$\eta(r_2) > l_1\left(\frac{cn}{(L(n))^{1/2}}\right) + \frac{r_1Ac^2}{8L(n)\,B}\,\eta(r_1)\,L\left(\frac{\eta(r_1)}{2B}\right).$$

Since L is a slowly changing function and

$$\eta(r_1) = l_1\left(\frac{cn}{(L(n))^{1/2}}\right) < 2B\psi^{-1}(r_1),$$

we have for $n > n_0$, $\psi((\log n)/3B) < r_1$; and

$$\eta(r_2) > l_1\left(\frac{cn}{(L(n))^{1/2}}\right) + \frac{\psi(\log n)\,Ac^2}{32B}\,\frac{1}{L(n)}\,l_1\left(\frac{cn}{(L(n))^{1/2}}\right)L\left(l_1\,\frac{cn}{(L(n))^{1/2}}\right),$$

and by (2.3) the last term on the right tends to ∞ with n. Hence for $n > n_0$ we have (3.6); that is,

$$\sup_{0<x<r_2}\left|\frac{1}{f(x)} - \frac{1}{p(x)}\right| > \frac{1}{2}\,\frac{1}{|\,p(r_2)|} > \exp\left(\frac{-2cn}{(L(n))^{1/2}}\right).$$

This leads to a contradiction with our assumption (3.2), since $c = \frac{1}{3}$. The theorem is proved.

Proof of Corollary 1.1. The proof is similar to that of Theorem 1. We take $f(z) = e_k(\eta(z))$, where $2 \leqslant k \leqslant K$; $\eta(r_1) = l_{k-1}(cn/L(n))^{1/2})$, $r_2 = r_1(1 + \delta/2)$, $\delta = c^2/2L(n)$. For $n > n_0$

$$\eta(r_2) > l_{k-1}\left(\frac{cn}{(L(n))^{1/2}}\right) + \frac{Ar_1}{L(n)}\,\psi^{-1}(r_1)\,L(\psi^{-1}(r_1)).$$

Now for the last term t (say) we have

$$t > \frac{A_1 r_1}{L(n)} \eta(r_1) L\left(\frac{\eta(r_1)}{2B}\right)$$

$$> \frac{A_2}{L(n)} \psi\left(l_{k-1} \frac{cn}{(L(n))^{1/2}}\right) l_{k-1}\left(\frac{cn}{(L(n))^{1/2}}\right) L\left(l_{k-1} \frac{cn}{(L(n))^{1/2}}\right)$$

since ψ, l_{k-1}, and L are all slowly changing functions. Hence for $k > 2$ (for the case $k = 2$, see Theorem 1),

$$t > \frac{A_2}{L(n)} L\left(l_{k-2} \frac{cn}{(L(n))^{1/2}}\right) \psi\left(l_{k-2} \frac{cn}{(L(n))^{1/2}}\right)$$

$$> \frac{A_2}{L(n)} L(l_{k-2}n) \psi(l_{k-2}n)$$

$$> A_1 \frac{L(n) \psi(n)}{(l_1 n) \cdots (l_{k-2}n) L(n)} .$$

Since

$$l_{k-1}\left(\frac{2cn}{(L(n))^{1/2}}\right) - l_{k-1}\left(\frac{cn}{(L(n))^{1/2}}\right) = (1 + o(1))(\log 2)(l_1 n \cdots l_{k-2}n)^{-1},$$

and $\psi(n) \to \infty$ we have for all $n > n_0$

$$\eta(r_2) > l_{k-1}\left(\frac{2cn}{(L(n))^{1/2}}\right),$$

and the rest of the proof is as in the theorem.

4. Proof of Theorem 2

Since

$$\psi(r) = \int_{r_0}^{r} \frac{dt}{tL(t)}$$

is a logarithmico-exponential function and increases to ∞, there exists an integer K such that $\psi(x)/l_K x$ tends to ∞ with x [3, p. 21]. Further $L(r)$ is also a logarithmico-exponential function [3, pp. 18–19]. We may suppose $K > 6$. Let $k = K - 3$ and $f(z) = e_{k+3}(z)$. Since $l_1 n l_2 n \cdots l_{k+2}(n)/L(n) \to \infty$ as $n \to \infty$ [3, pp. 33–34] the argument given in Theorem 1 completes the proof.

5. Proof of Theorem 3

Let $p \in \pi_n$ be a polynomial such that

$$\lambda_{0,n} = \sup_{0<x<\infty} \left| \frac{1}{f(x)} - \frac{1}{p(x)} \right|$$

Suppose first that $1/\lambda_{0,n} > f(d_n)$ $(n \geqslant 1)$. Then since f is increasing on $[0, \infty)$ we have for $0 \leqslant x \leqslant d_n$

$$|f(x) - p(x)| \leqslant \frac{\{f(d_n)\}^2 \lambda_{0,n}}{1 - (\lambda_{0,n}) f(d_n)} . \tag{5.1}$$

Now write

$$E_n(f) = \inf_{g \in \pi_n} \| f - g \|_{[0,d_n]} .$$

Then

$$E_n(f) \leqslant \| f - p \|_{[0,d_n]} \leqslant \{f(d_n)\}^2 \Big/ \left\{ \frac{1}{\lambda_{0,n}} - f(d_n) \right\} . \tag{5.2}$$

By an inequality of Bernstein [1, p. 10]

$$E_n(f) \geqslant 2(a_{n+1})(d_n/4)^{n+1}. \tag{5.3}$$

From (5.2) and (5.3) we get, for $n \geqslant 1$,

$$\lambda_{0,n} > \frac{a_{n+1} d_n^{n+1}}{2^{2n+2}\{f(d_n)\}^2} . \tag{5.4}$$

If $1/\lambda_{0,n} \leqslant f(d_n)$ then since $f(d_n) \geqslant a_{n+1} d_n^{n+1}$, (5.4) certainly holds. The proof is complete.

Proof of Corollary 3.1. By the argument of Theorem 3, we have

$$\lambda_{0,2n-1} \geqslant (a_{2n} d_n^{2n})/(2^{4n}\{f(d_n)\}^2). \tag{5.5}$$

Now, $f(z)$ can be written as

$$f(z) = a_0 + \sum_{k=1}^{\infty} a_k z^k = a_0 + a_0 \sum_{k=1}^{\infty} \frac{z^k}{d_1 \cdots d_k} .$$

Let $\mu(r, f)$ denote the maximum term and $\nu(r, f)$ its rank. Then for $r > r_0$ [12]

$$M(r, f) < A_1 \mu(r, f)\, \nu(r, f)$$

and we can choose A_1 such that this inequality holds for all $r \geqslant d_1$. This gives

$$f(d_n) < a_0 A_1 n d_n{}^n/(d_1 \cdots d_n) \tag{5.6}$$

and (2.12) follows from (2.9), (5.5), and (5.6), on writing $A = 1/a_0 A_1{}^2$.

Proof of Corollary 3.2. We have for $r > r_0$ [6, p. 498]

$$M(r,f) < A_2 \mu(r,f)\, \nu(r,f)$$

and now the argument is similar to that in Corollary 3.1.

Proof of Corollary 3.3. By Theorem 3, we get for $n \geqslant 1$,

$$\lambda_{0,2n-1} \geqslant a_{2n} D_n^{2n}/2^{4n}\{f(D_n)\}^2.$$

Here $\{D_n\}_1^\infty$ is any strictly increasing sequence, $D_1 > 0$. Now we can write

$$f(z) = a_0 + \sum_{k=m_0}^{\infty} \frac{z^k}{d_1 d_2 \cdots d_k},$$

where $d_{k+1} > d_k$ for $k > N_0$. Choose n_0 ($>N_0$) so large that for $n \geqslant n_0$,

$$d_n > \max_{1 \leqslant k < n} d_k .$$

Take $D_n = d_n$, $n \geqslant n_0$, and $0 < D_1 < D_2 < \cdots D(n_0) < \cdots$. Then, for $n > n_0$

$$\lambda_{0,2n-1} \geqslant \exp\{\log a_{2n} + 2n \log d_n - 4n \log 2 - 2 \log f(d_n)\}.$$

Now [8, p. 9] $\log M(r) \sim \log \mu(r) \sim r^{\rho(r)}$, and $\nu(r) \sim \rho r^\rho L(r)$ (cf. [3, p. 38]). Hence $n = (1 + o(1))\, d_n{}^\rho L(d_n)$. Further $(\rho(r) - \rho) \log r = \log L(r)$, $r\rho'(r) \log r = o(1)$. Hence

$$\log \frac{L(d_{2n})}{L(d_n)} = o\left(\log \frac{d_{2n}}{d_n}\right),$$

$$\log \frac{d_{2n}}{d_n} = \frac{1}{\rho}(\log 2 + o(1)).$$

Consequently,

$$\begin{aligned}\lambda_{0,2n-1} &\geqslant \exp\left\{\log \frac{d_n^{2n}}{d_1 d_2 \cdots d_{2n}} - 2 \log f(d_n) - 4n \log 2\right\} \\ &= \exp\left\{d_n^{2\rho} L(d_{2n}) - \frac{2n}{\rho}(\log 2) - 2 d_n{}^\rho L(d_n) - 4n \log 2 + o(n)\right\} \\ &= \exp\left\{-4n \log 2 - \frac{2n \log 2}{\rho} + o(n)\right\}.\end{aligned}$$

Since $\lambda_{0,2n-2} \geqslant \lambda_{0,2n-1}$, the corollary follows.

6. Proof of Theorem 4

(i) We compare the growth of

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

with the growth of

$$F(z) = a_0(F) + \sum_{k=n_0+1}^{\infty} \left(\frac{1}{kL(k)}\right)^{k/\rho} z^k \equiv \sum_{k=0}^{\infty} a_k(F)\, z^k, \quad a_0(F) > 0. \tag{6.1}$$

By hypothesis on a_k we have for $r > r_0$

$$\frac{M(r, F)}{2} < M(r, f) < 2M(r, F). \tag{6.2}$$

Further F, and so f, are of perfectly regular growth with respect to a proximate order $\rho(r)$ defined as follows [14, pp. 209–211]. Let $x > n_0$ and

$$w(x) = \{\log(xL(x))\}/(\rho \log(x/e) - \rho \log \rho).$$

Then

$$\lim_{x\to\infty} w(x) = 1/\rho, \qquad \lim_{x\to\infty} w'(x)\, x \log x = 0.$$

Let $\rho(r) = 1/w(y)$, where $r = \{yw(y)\}^{w(y)}$. Then $\rho(r)$ is a proximate order, $\lim_{r\to\infty} \rho(r) = \rho$ and

$$\lim_{r\to\infty} \frac{\log M(r, F)}{r^{\rho(r)}} = \lim_{r\to\infty} \frac{\log M(r, f)}{r^{\rho(r)}} = 1.$$

(ii) We now prove (2.15). Choose

$$d_n = \frac{a_{n-1}(F)}{a_n(F)}, \qquad n > n_0 .$$

Then by convexity hypothesis

$$d_n < d_{n+1} .$$

Also for $n > n_0$, $d_n \geqslant d_p(n/(n-1))$, $p = [n/2]$. Hence [12] $F(d_n) < An\, a_n(F)\, d_n{}^n$. Consequently we have by (5.5), for $n > n_0$,

$$\lambda_{0,2n-1}^{1/(2n-1)} \geqslant \exp \left\{\frac{1}{2n-1} (\log a_{2n} + 2n \log d_n - 4n \log 2 - 2 \log f(d_n))\right\} . \tag{6.3}$$

Now

$$2n \log d_n - 2 \log f(d_n) \geqslant 2n \log d_n - 2 \log a_n(F) - 2n \log d_n + o(n),$$

$$\log \frac{a_{2n}}{\{a_n(F)\}^2} = o(n) - \frac{2n}{\rho} \log 2.$$

Hence

$$\liminf_{n\to\infty} \lambda_{0,2n-1}^{1/(2n-1)} \geqslant \exp \left\{ -2 \log 2 - \frac{1}{\rho} \log 2 \right\},$$

and (2.15) is easily proved.

(iii) To prove (2.14), we note that

$$0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)} \leqslant \frac{\sum_{k=n+1}^{\infty} a_k x^k}{s_n^2(x)}.$$

Now let $n > n_0$. Using convexity hypothesis we have

$$\sum_{k=n+1}^{\infty} a_k x^k < 2 \sum_{k=n+1}^{\infty} \left(\frac{1}{kL(k)} \right)^{k/\rho} x^k$$

$$< 2x^{n+1} \left(\frac{1}{(n+1) L(n+1)} \right)^{(n+1)/\rho} (1 + T_1 + T_1^2 + \cdots)$$

$$= \frac{2x^{n+1}}{1 - T_1} \left(\frac{1}{(n+1) L(n+1)} \right)^{(n+1)/\rho},$$

where

$$T_1 = \frac{x((n+1) L(n+1))^{(n+1)/\rho}}{((n+2) L(n+2))^{(n+2)/\rho}} = \frac{x}{\chi(n)} < 1.$$

Suppose now n is odd, $n = 2N - 1$. Then $\{s_n(x)\}^2 \geqslant a_N{}^2 x^{2N}$ and so

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} < \frac{2}{a_N{}^2 \{2NL(2N)\}^{(2N/\rho)}} \frac{1}{1 - T_1}.$$

Choose $x \leqslant \chi(n)(1 - \delta_n)$, where $\delta_n = \exp(-n/\log n)$. Then

$$\frac{1}{1 - T_1} \leqslant \frac{1}{\delta_n},$$

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} < \exp \left\{ o(n) + \frac{2N}{\rho} \log(NL(N)) - \frac{2N}{\rho} (\log(2N) + \log L(2N)) \right\}$$

$$= \exp \left\{ \frac{-2N}{\rho} \log 2 + o(N) \right\}. \tag{6.4}$$

Now if $x > \chi(n)(1 - \delta_n)$, then

$$0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)} \leqslant \frac{1}{a_N x^N}$$
$$= \exp\left\{\frac{N}{\rho}\log(NL(N)) + O(1) - N\log\chi(2N-1) - N\log(1-\delta_{2N-1})\right\}$$
$$= \exp\left\{\frac{-N}{\rho}(1+\log 2) + o(N)\right\}. \tag{6.5}$$

Hence for all large N, the expression in (6.5) is less than (6.4). Consequently,

$$\limsup_{N\to\infty} \lambda_{0,2N-1}^{1/2N-1} \leqslant \exp\left(\frac{-1}{\rho}\log 2\right).$$

Also $\lambda_{0,2N} \leqslant \lambda_{0,2N-1}$. Hence from (6.4)

$$\limsup_{N\to\infty} \lambda_{0,2N}^{1/2N} \leqslant \exp\left(\frac{-1}{\rho}\log 2\right)$$

and the theorem is proved.

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